Mechanics of the Compression Wood Response

I. PRELIMINARY ANALYSES

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ABSTRACT

Righting of two tilted white pine (Pinus strobus L.) stem leaders by compression wood formation was followed for 16 weeks. The natural curves and three deflection curves under added end loads were determined from weekly field photographs. Data for self-loading and cross sectional diameters were interpolated from original estimated and final measurements. A mechanical-mathematical model was developed to predict curves under zero gravity for each stem each week. The model estimated stiffness of the leaders independently for each week, and the stiffnesses were consistent throughout the experiment. A second model was developed to simulate the deflection curves assumed when the zero gravity curves were subjected to different end loads. These predicted curves were nearly identical to the observed curves from the photographs, thus verifying the assumptions in the first model. Data from this study will be used to investigate the mechanical aspects of compression wood induction and action as the stem is bent upward toward the vertical.

The induction of compression wood formation appears to be a response to disorientation in a gravitational field leading to various mechanical effects. The action of compression wood results in the mechanical bending of a tilted stem. A great deal of work has been done on the more biological aspects of compression wood over the years, but not very much on the mechanical aspects (9, 10), in part because the analysis of the nonlinear bending of tapered rods has only recently been practical using modern high speed computers. We have chosen to analyze a relatively simple example of compression wood induction and action, the erection of tilted, young pine stems. Westing has performed a similar experiment to investigate a different aspect of the compression wood response (8).

The analysis of this simple experiment to obtain the stress distribution and changes in slope during righting requires periodic data on the shape, stiffness, cross sectional dimensions (including distribution of compression wood), and orientation to the vertical of the stem as it is erected during the course of a season of compression wood formation. This paper reports the mathematical models and computer techniques developed to analyze the shapes and stiffness of the leaders from simple, nondestructive, field photos and data taken during the experiment, combined with cross sectional data taken after the experiment was over. Another paper will report on the correlation among orientation to vertical, stress distribution, and compression wood formation and on the mechanical strains developed by compression wood as it erects the tilted stems.

MATERIALS AND METHODS

Data Collection. On May 16, 1969, two fast growing, 3.5 m tall, white pine trees (Pinus strobus L.) in a field near Amherst, Massachusetts, were bent over and attached to stakes just below the 1968 whorl (Fig. 1) so that the 1968 leaders (67 and 77 cm long) were tilted at, respectively, 30° and 90° from the vertical. Four side view photographs were taken of each leader each week from May 16 to September 4, 1969. Photos were taken from the same point and included the stake for vertical orientation and scale. Each set of photos included the unweighted leader and the leader with three successively heavier weights hung by a tape collar at the end of the 1968 leader (Fig. 1) for just long enough to take the photo. The weights were selected to produce visible changes in shape of the leader. They were 50, 100, and 200 g until June 6, when they were increased to 100, 200, and 300 g because of increased stiffness of the leaders. As the 30° leader became more vertical, the weights were further increased about every 2 weeks until by August 22 they were 500, 1000, and 1500 g. The lengths of the new, 1969 leaders and the vertical and horizontal diameters at six points on each 1968 leader were measured weekly.

Two trees, comparable to the tilted trees, were cut on May 16 to determine the fresh weight of the 1969 leader and 1968 whorl of branches (total about 50 g) and the density of the 1967 and 1968 leaders (about 1.0 g/cc). On September 4 the experimental trees were cut. The 1969 leaders and whorls and the 1968 leaders were weighed. The 1968 leaders were then cut transversely into 2.5-cm segments. The cut surfaces were smoothed with a sliding microtome, and drawings of the cross sections were made (×7.2) with a Wild stereomicroscope and drawing tube to outline the pith, first annual ring, extent of compression wood, and the outer boundaries of the wood and bark (Fig. 2). Diameters of each segment were measured at the 17 points on these drawings shown in Figure 2.

The weekly external diameter measurements showed that thickening of the 1968 leader was essentially linear throughout the experimental period, and so diameter estimates for each segment were interpolated linearly between the beginning of the
experiment, marked in the cross section by the first formed compression wood, and the diameters at the time of cutting. The 1969 leaders stopped elongating before September 4 (70 cm on July 18 for the 30° and 25 cm on June 12 for the 90°), but weight increase due to cambial activity presumably continued on as in the 1968 leaders, so we assumed that the weight increase of the 1969 leaders and whorls was linear with time from the initial 50 g to the final weight of 230 g for the 30° and 136 g for the 90°.

Tests of 1- and 2-year-old leaders with and without bark showed that although bark comprises about a fourth of the total diameter of young leaders, it is so elastic that it can be disregarded in our stiffness analyses.

The $x, y$ coordinate data on scale, vertical orientation, and shape of the leaders were converted to computer-readable form directly from 20-×25-cm enlargements of the weekly photographs, using automatic coordinate recording tablet equipment.

**Analytical Techniques.** A series of computer programs was developed which interpreted the photographs and section data and simulated the growing stem. By means of a continuum mechanical model which predicts the large deflection of a tapered elastic rod under a given set of loads, it was possible to extrapolate from the experimental deflection shapes to obtain the zero gravity curve of the stem as it responded to the eccentric gravitational loads. Also an effective elastic modulus in bending for the growing stem was calculated as part of the same analysis for each series of experiments over the season.

It will be seen that the results are remarkably self-consistent considering the inevitable lack of precise information on some of the parameters entering the problem.

**Mechanical Model.** For mechanical simulation purposes the stem is assumed to be a homogeneous tapered cantilevered rod fixed to the stake at the end A and subjected to combinations of known weights at B as shown in Figure 1. In subsequent studies a composite rod with different moduli for “normal” wood and compression wood will be analyzed. The nonlinear bending theory of inextensible thin rods is well known in engineering mechanics. Using the notation of Mitchell (4), let the “zero gravity” shape of the neutral curve (defined as the set of points in the plane parallel to the $xy$ plane which divides the rod sections into two symmetric parts and where the bending stress is zero) of the rod be given by

$$\psi = \psi(s)$$

and the loaded shape by

$$\phi = \phi(s)$$

as shown in Figure 3. Since the rod is assumed to be inextensible, each point on the loaded or unloaded rod is uniquely located by the arc length coordinate $s$, measured along the neutral curve, and the slope angles $\psi$ and $\phi$ completely specify the shape of the rod (provided the rod bends in the original $xy$ plane which was found to be substantially the case). Using the hypothesis that the bending moment at each point is linearly related to the change in curvature, this gives

$$M(s) = EI(s) \left( \frac{d\phi}{ds} - \frac{d\psi}{ds} \right)$$

where $E$ is Young's modulus, $I(s)$ is the moment of inertia of the cross section, and $M$ is the bending moment. It may be noted...
here that $E$, $I$, and $\psi$ are all taken as functions of time over the growing season.

By considering the moment equilibrium of the Section QB of the rod, it follows that

$$\frac{dM}{ds} - V(s) \cos \phi - H(s) \sin \phi = 0$$

where

$$H(s) = \int_s^L h(s_1) ds_1$$

$$V(s) = P_0 + \int_s^L v(s_1) ds_1$$

$L$ is the length of the stem, $P_0$ is the end load at B (Fig. 3), and $h(s)$ and $v(s)$ are the distributed loads in the horizontal and vertical directions per unit of length.

For the set of experiments studied in this paper, the $Y$ direction is carefully chosen to coincide with the vertical so that no horizontal loads are acting; thus, $H(s) = 0$.

**Determination of Zero Gravity Shapes.** The usual situation in engineering applications of the above theory is to be given the shape $\psi(s)$, the modulus $E$, the cross sectional geometry from which $I(s)$, the moment of inertia, is computed, and a given set of loads; and to be required to predict the response, or new shape, $\psi(s)$. However, for botanical problems in which the cross sectional geometry is being continuously modified by cambial activity, as is the case here, we must solve an inverse problem.

The cross sectional geometry was estimated and $I(s)$ derived as described in Appendix A. The "end loads" due to the 1969 leader and 1969 whorl at B and the weight at the 27 sections (90°) and 30 sections (0°) along the stem were estimated for any given day during the experiment.

An approximation of the natural shape is estimated from the following information:

a. A set of coordinates $(x_k, y_k)$, $k = 1, \ldots, M$, of the center line of the loaded stem, together with scale information. Actually the coordinates of the neutral axis (zero bending stress) are desired, but it is assumed that these points lie close to the center line.

b. A set of measurements for cross sectional configurations of pith diameter, major and minor axes of the "xylem ellipse" (see Appendix A), and the bark thickness—interpolated for any day in the course of the experiments.

c. A set of measurements of the end loads from 1969 leader and whorl interpolated for any day during the experiment.

The mathematical steps required in order to obtain $\psi(s)$ are as follows:

a. Compute $V(s)$ from equation 6 using

$$v(s) = \rho g A(s)$$

where $\rho g$ is the weight density of the stem material, $A(s)$ is the variable cross sectional area and

$$P_0 = P_1 + P_s$$

b. Compute $M(s)$ from equation 4 by integration:

$$M(s) = -M_0 + \int_s^L V(s_1) \cos \phi(s_1) ds_1$$

where $M_0$ is the moment of the living material beyond B acting on the stem at B. $\phi(s)$ is the slope of the stem as computed from the coordinates of the center line in the deformed state as measured from photos.

c. Compute $\psi(s)$ from equation 3 by integration:

$$\psi(s) = \psi(s) - \int_s^L \frac{M(s)}{EI(s)} ds_1$$

where $E$ is Young's modulus for the stem material and is assumed to be a constant with respect to location along the stem. $I(s)$ is the moment of inertia of the cross section about the neutral axis in bending.

If one had exact values as to cross sectional geometry, density, deformed geometry, material properties, etc., then from a single known applied load at B, a zero gravity shape, $\psi(s)$, could be precisely computed by the above steps. Within the assumptions of the theory for nonlinear deflection of rods, the zero gravity shape would be uniquely determined.

In the present case, it seemed appropriate to carry out multiple tests and combine the predictions of the zero gravity shape for the four different loaded cases to get the best prediction in the least squares sense.

The tests were considered in pairs. Consider tests $k$ and $l$, then

$$\psi_i(s) = \psi_i(s) - \frac{1}{E} \int_s^L \frac{M_i(s)}{I_i(s)} ds_1$$

where the subscript of a variable denotes that it was computed with the information from that test. Assume that $E$ and $I(s_l)$ do not vary for a given set of tests on a given day.

Since $\psi(s)$ and $E$ are unknown and assumed to be common to both tests, we chose $E$ so that

$$R_k(E) = \sum_N [\psi_k(s_1) - \psi_l(s_1)]^2 = \text{minimum}$$

where $s_1$ are a selected set of points over the length.

Let

$$G_k(s) = \int_s^L \frac{M_k(s)}{I_k(s)} ds_1$$

then

$$R_k(E) = \sum_N [\psi_k(s_1) - \psi_l(s_1)]^2 = \sum_N [G_k(s_1) - G_l(s_1)]$$

A simple extension of this analysis which treats all pairs of tests independently in the least squares computation for $E^{-2}$ is possible. For example, with three tests let

$$S = R_{12} + R_{13} + R_{3a}$$

then writing

$$H_{12a} = G(s_1) = G(s_2)$$

$$Z_{12a} = \psi(s_1) - \psi(s_2)$$

$$\frac{\partial S}{\partial E^{-2}} = \frac{\partial R_{12}}{\partial E^{-2}} + \frac{\partial R_{13}}{\partial E^{-2}} + \frac{\partial R_{3a}}{\partial E^{-2}} = 0$$

or

$$\frac{\partial S}{\partial E^{-2}} = \left[ \sum_N Z_{12a} H_{12a} + Z_{13a} H_{13a} + Z_{3a} H_{3a} \right] = 0$$

For most of the experiments, four cases were available for each stem so that six terms were used in the above analysis.

Using the "optimal" $E^{-2}$ value, an average curve for $\psi(s)$ is computed from

$$\psi(s) = \frac{1}{N} \sum_N [\psi(s_1) - E^{-2} G(s_1)]$$

where $N$ is the number of independent pairs and $i = 1, N$. 
Thus one obtains tabular values for $\psi(s)$ and an optimal $E$ value.

The $x$, $y$ coordinates of the natural curve can be found from the relations

$$x(s) = \int_0^s \cos \psi(s) \, ds$$

$$y(s) = \int_0^s \sin \psi(s) \, ds$$  \hspace{1cm} (22)

$$y(s) = \int_0^s \sin \psi(s) \, ds$$  \hspace{1cm} (23)

Simulation of the Mechanical Response to Applied Loads. From the results of the previous section, we have the zero gravity shape $\psi(s_1, t_1)$ and the bending modulus $E(t_2)$ where the $s_1$ are a set of known locations along the stem, and the $t_2$ are the sampling dates of the experiment. If the assumptions as to the cross sectional geometry, end loads, constancy of the bending modulus along the stem, etc., are reasonably self-consistent, then mechanical responses predicted by the model on any given day should give good agreement with the observed response.

The direct problem (as contrasted with the inverse problem of the previous section) consists of solving a nonlinear boundary value problem to find the predicted deflection resulting from a known applied load.

A differential equation for the unknown response $\phi(s)$ to given loading can be derived by substituting $M(s)$ from equation 3 into equation 4 to get

$$\frac{d}{ds} \left[ EI(s) \left( \frac{d\phi}{ds} - \frac{d\psi}{ds} \right) \right] = V(s) \cos \phi = 0$$  \hspace{1cm} (24)

In addition to equation 22, $\phi$ must satisfy the boundary conditions:

$$s = 0: \phi(0) = \psi(0)$$  \hspace{1cm} (25)

$$s = L: \frac{d\phi}{ds} = \frac{d\psi(L)}{ds} - \frac{M_A}{EI(L)}$$  \hspace{1cm} (26)

Equation 23 states that the slope at the fixed end (A) must coincide with the zero gravity shape, and equation 24 states that the bending moment at B (expressed in terms of $\phi$) must equal the prescribed moment, $-M_A$, which is generated by the living material beyond B as well as any moment caused by the additional weights applied in the various cases.

Thus the prediction of the response of the stem to an applied load on a specified day would be carried out as follows: (a) $V(s)$ is computed as in equations 7 and 8. (b) $\phi(s)$ is found for a specified $E$, $\psi(s)$, and $I(s)$ by solving equations 24 to 26. (See Appendix B for a discussion of the technique for the numerical solution of equations 24-26.)

Comparison of these predicted responses with the raw data from the photos should give an indication of the self-consistency of the analysis leading to the zero gravity shape.

RESULTS

Zero Gravity Shapes and Elastic Moduli. Forty to 45 data points were available for the description of each deflection curve. The $\phi(s)$ curve used in equations 9 and 10 was computed by means of a sliding quadratic smoothing process. (See Appendix C for details.) A quadratic (second degree) curve with three unspecified parameters was fitted to five consecutive points by least squares. The slope $dy/dx$ at the midpoint was then estimated from the value of the derivative of the quadratic evaluated at the midpoint (off center formulae were used near the first and last points). $\phi(s)$ can be found from

$$\phi(s) = \tan^{-1} \left( \frac{dy}{dx} \right)$$  \hspace{1cm} (27)

For the integrations required in equations 6, 9, and 10, a simple trapezoidal rule turned out to give adequate accuracy.

The results for the effective elastic modulus are presented as computed using equation 20 in Figure 4. The average root mean square error between the $\psi(s)$ computed from equation 21 and the $N_i$ individually predicted $\psi$ curves ranged from 0.5 to 2.0% of slope. In Figures 5 and 6 the variation of the natural stem curve (i.e., subjected only to natural gravity loads) with date is shown. Note that the slope at A is substantially constant; thus the assumption of no rotation of the stem about A is verified.

Considering the wide range of loading and the extensive change in the natural shape of the stem over the season, the variation shown in the effective modulus $E$ is not surprising, especially since it has been reported (see Ref. 10, p. 453) that the modulus of elasticity in compression for green compression wood is about 67% of the value for normal wood for ponderosa pine (Pinus ponderosa Laws.). Since the $E$ computed here actually represents an effective equivalent modulus for the combined effect of the two types of wood in the section and since the fraction of the section which contains compression wood is changing along the stem and with time, it is not surprising that $E$ varies significantly with time. Further work will treat the stem as a composite rod made up of two materials with separate bending moduli.

Stem Deflection Model. It is now possible to check the consistency of the analysis which determined the zero gravity shape and elastic modulus by carrying out a simulation of the mechanical loading of the stem. This requires the solution of equation 24 as outlined before.

The iterative scheme and computer program described in Appendix B were checked by running the case of an initially straight rod, of constant cross section both for end load and for uniformly distributed loading, and comparing the results with published solutions. The numerical solution, presented above, resulted in a vertical deflection and end slope less than 1% different from the results of Khwaja (3), who used an analogue computer to solve the end-loaded cantilever, or from the exact solution of Frish-Fay (2). For the uniformly loaded cantilever, Wang et al. (7) have given a graph (Fig. 3 in that paper) of their results, and the present results coincide with their results at four different load levels.

Results for the simulations of the loaded stem are shown in Figures 7 and 8. In each case the solid curves indicate the predicted responses based on theory (i.e., integration of equations 24-26). Agreement with the raw data (crosses) from the photographs is very good. Results for other dates and loads followed along similar lines with about the same range of accuracies of fit as shown in Figures 7 and 8.

CONCLUSIONS

We have established the feasibility of the technique of constructing a mathematical-mechanical model of the compression
Figs. 5 and 6. Natural shape (only natural gravity loads) of the two 1968 leaders at biweekly intervals through the experimental period taken from photographs. The leaders first saged and then were bent up by compression wood action. Differences in length are due to inaccuracies in marking A and B on the photos. In the models all lengths were corrected to the measured final lengths. Crosses are raw data and solid lines were computed by the smoothing procedure (Appendix C).

Figs. 7 and 8. Examples of measured deflection curves (crosses) compared to points on deflection curves predicted by the mechanical-mathematical models (solid lines) from the calculated zero gravity curves (dotted lines). ZG: Zero gravity curve; 0g-100g: weight in grams of added end load at point B.

wood response by processing geometric data directly from field photographs. We have also confirmed that the predicted elastic moduli and zero gravity curves used in the model are realistic when compared with actual experimental load responses.

For further quantitative study of the compression wood response our analyses have provided the following information.

1. Values for the slope angle (inclination from the vertical) along the entire length of both test stems as a function of time over the 16-week experimental period. The slope angle is a measure of the geotropic portion of the compression wood stimulus.

2. Values for stress levels at any point in the cross section of the stem at points along the length of the stem as a function of time over the growth period. These values can be used to test the possible role of stress in compression wood stimulation.

3. Changes in curvature along each stem over the experimental period. These changes can be used to compute the strains, from which the magnitude of mechanical action of the compression wood can be determined.

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LITERATURE CITED


APPENDIX A

Simulation of a Growing Section. To obtain estimates for (a) the area of the cross section (as far as contributions to the
weight/unit length) and (b) the moment of inertia about a horizontal line through the centroid of the section including those parts of the section which significantly contribute to the effective stiffness of the section, we computed daily “xylem ellipses” which were fitted to the interpolated section geometry.

In Figure 2 major and minor axes \(a_t\) and \(b_t\) are indicated for an ellipse which approximates the xylem at the beginning of the tilting experiment \((t = 0\), where \(t\) measures days elapsed in the experiment). Let

\[
\begin{align*}
\beta &= b_t(1 + c_t) \\
\alpha &= a_t(1 + c_t)
\end{align*}
\]

where \(b_t\) and \(b\) are the radial distances from \(O\), the origin of the \(n, z\) coordinate system, to the highest and lowest points on the current xylem; and \(2a\) is the width of the current xylem. The growth rates \(c_t\), \(c\), and \(c\) are computed by requiring that for

\[
\begin{align*}
\gamma(t) &= b_t \quad \text{or} \\
\gamma(t) &= \frac{[b_t/b] - 1]}{t_f} \\
\gamma(t) &= \frac{[b_t/b] - 1]}{t_f} \\
\gamma(t) &= \frac{[b_t/b] - 1]}{t_f}
\end{align*}
\]

If it is assumed that an ellipse gives a reasonably good fit for any \(t\) \((0 \leq t \leq t_f)\) then take

\[
\begin{align*}
\gamma(n/b) \gamma - 2\gamma(z/a) + \gamma + (z/a) = 0
\end{align*}
\]

where the \(\gamma\) are functions of \(t\) and \(n\) and \(z\), the vertical and horizontal coordinates.

The \(c_1\), \(c_2\), \(c_3\) are determined at any \(t\) from:

\[
\begin{align*}
z &= 0; n = b_1, \quad n = b_2 \\
\gamma &= 0; z = a
\end{align*}
\]

Note that for \(t = 0\), we have \(c_2 = -1\), \(c_1 = 1\) and \(c_3 = 0\).

In general, equation A-12 leads to

\[
\begin{align*}
\gamma(n/b) \gamma - 2\gamma(z/a) + \gamma + (z/a) = 0
\end{align*}
\]

where \(\gamma(n/b) \gamma - 2\gamma(z/a) + \gamma + (z/a) = 0\)

Finally, the area to be used in the weight per unit length calculation (equation 7) includes the pith and the bark.

\[
A = \pi(d + b_0)(b_0 + b_0)
\]

where \(b_0\) is the average bark thickness calculated from the four thickness measurements at the coordinate axes.

**APPENDIX B**

Iterative Solution of Nonlinear Boundary Value Problem. Define

\[
\beta = \phi - \psi
\]

then equations 24 to 26 become

\[
(D\beta')' - V \cos(\psi + \beta) = 0
\]

and

\[
s = 0: \beta = 0
\]

Finally, the area to be used in the weight per unit length calculation (equation 7) includes the pith and the bark.

\[
A = \pi(d + b_0)(b_0 + b_0)
\]

where \(b_0\) is the average bark thickness calculated from the four thickness measurements at the coordinate axes.

Now if \(\beta(s)\) is an estimate of the solution of equations B-2 to B-4, we write the actual solution \(\bar{\beta}\) as

\[
\beta = \bar{\beta} + \delta\beta
\]

where \(\delta\beta\) is assumed in general to be a small correction to \(\bar{\beta}\). (If the process turns out to be convergent this would be the case.) Expanding the cos term with \(\delta\beta\) small gives

\[
\cos(\phi + \beta) = \cos(\phi + \bar{\beta}) - \delta\beta \sin(\phi + \bar{\beta}) + \cdots
\]

Stopping with a linear term in \(\delta\beta\), it follows that

\[
\cos(\phi + \beta) = \cos(\phi + \bar{\beta}) - (\beta - \bar{\beta}) \sin(\phi + \bar{\beta})
\]

Thus equation B-2 can be written as

\[
[D(s)\beta'] + f(s)\delta = g(s)
\]

where

\[
f(s) = K(s) \sin(\phi + \beta)
\]

Now equation B-10 was solved for any estimate \(\beta\) by replacing the differential equation by a finite difference version and solving simultaneous equations for \(\beta\) at a discrete set of points.

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2 See Reference 5 for a more complete discussion of this method.
For the present problem the unknowns are \( m + 1 \) values of \( \beta \), say \( \beta_k = \beta(s_k) \), at the points \( s_k = k\Delta s, k = 0, \ldots, m \) and \( \Delta s = L/m \). If equation B-10 is written in finite difference form as

\[
\frac{\delta}{\Delta s} \left[ D_k \frac{\delta f_k}{\Delta s} \right] + f_k \delta g_k = g_k \quad (k = 1, \ldots, m - 1)
\]

(B-13)

where

\[
\delta(\cdots)_k = (\cdots)_{k+1/2} - (\cdots)_{k-1/2}
\]

and \( D_k, f_k, \) and \( g_k \) are known values at the \( s_k \) points.

The boundary conditions give

\[
\beta_0 = 0 \\
\frac{\delta \beta_m}{\Delta s} = -M_m/D_m
\]

(B-15)

(B-16)

Now equations B-13, B-15, and B-16 represent \( m + 1 \) equations for \( m + 1 \) unknowns \( \beta_k \). Since equation B-13 only couples each unknown \( \beta_k \) to, at most, its neighbors on each side, \( \beta_{k+1} \) and \( \beta_{k-1} \), these simultaneous equations can be rapidly solved by tridiagonal matrix inversion techniques (1).

For the results such as those shown in Figures 7 and 8 the following steps were followed:

a. Forty-one values of \( \phi \), for a given stem from the zero gravity shape determination as well as the modulus \( E \) were used to set up the \( f(s_k) \) and \( g(s_k) \). The \( V_k \) and \( E I_k \) were obtained from known data.

b. For the first trial \( \beta_k \) was taken as 0, and usually one to three iterations lead to convergence where in each case the solution of equation B-10 was "fed back" as the new \( \beta \) until no significant change in \( \beta \) for the given load level was found.

**APPENDIX C**

**Slope Determination by a Sliding Quadratic Smoother.** It was necessary to smooth the data from field photographs before computing the slope \( \phi(s) \). A simple 5-point quadratic smoother was found to be appropriate.

Consider the \( k^{th} \) point \( (k = 3, \ldots, N_p - 2) \) where \( N_p \) is the number of centerline coordinates, define the quadratic

\[
y = \alpha_0 + \alpha_1 x + \alpha_2 x^2
\]

(C-1)

Let the \( \alpha_1, \alpha_2, \alpha_3 \) be determined so that

\[
F_k = \sum_{m=2}^{m-1} [\alpha_0 + \alpha_1 \tilde{x}_m + \alpha_2 \tilde{x}_m^2 - \tilde{y}_k]^2
\]

(C-2)

is a minimum where \( (\tilde{x}_k, \tilde{y}_k) \) are coordinates of the raw data. The usual least squares analysis then computes \( \alpha_0, \alpha_1, \) and \( \alpha_2 \) from the three equations found by setting the derivatives of \( F_k \) with respect to the \( \alpha \)'s equal to zero.

The smoothed values of \( y \) and \( dy/dx \) at the \( k^{th} \) point follow as

\[
y(\tilde{x}_k) = \alpha_0 + \alpha_1 \tilde{x}_k + \alpha_2 \tilde{x}_k^2
\]

(C-3)

\[
\frac{dy}{dx}(\tilde{x}_k) = \alpha_1 + 2\alpha_2 \tilde{x}_k
\]

(C-4)

Also

\[
\phi(\tilde{x}_k) = \tan^{-1} \left[ \frac{dy}{dx}(\tilde{x}_k) \right]
\]

(C-5)

where the arc length \( s_k \) is found from the integral

\[
s_k = \int_{x_0}^{x_k} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

(C-6)

where equation C-6 is evaluated by the trapezoidal rule using smoothed values for the \( dy/dx \) in the integrand.

For the cases \( k = 1, 2, N_p - 1, N_p \) special "end point type" smoothing procedures similar to the one just described were used.

For stem configurations which were highly vertical, a version of the above procedure which reversed the roles of \( x \) and \( y \) was used. For intermediate cases where neither vertical nor horizontal tangent lines were involved, it was possible to get an excellent check on the self-consistency of the smoothing procedure by using both the \( x \) versus \( y \) and \( y \) versus \( x \) smoothing. Comparison of the results showed very similar results for the coordinate invariant form \( \phi(s) \) of the middle-line curves. In Figures 5 and 6 both the raw data points and the smoothed coordinates of the stem center line are shown.